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Solving a Non-Linear Constrained Portfolio Optimization Problem; Applications of Lagrange Kuhn-Tucker Method[1]

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Abstract

Individual or corporate investors try to create a portfolio that minimizes risk and maximizes returns when investing in stocks. One of the most beautiful models that make portfolio selection according to these conditions is Markowitz's mean–variance model. By using this model, we constitute optimal portfolios from 30 stocks strongest capitals in Turkey. The aim of this study is to find the weight of the stocks to be invested in the optimum portfolio that is created, that is, to calculate how much investment should be made to which stock. In this case, as it is expected, the problem of non-linear constrained portfolio optimization with single-objective function is obtained. In this paper, the weights of the stocks that make up the optimal portfolios are solved by using the Kuhn-Tucker method and Matlab programming language rather than the traditional methods.

Keywords: BIST 30, Expected return, Kuhn-Tucker method, portfolio optimization.

1. Introduction

A portfolio is an asset formed by a combination of a series of securities for a purpose (Huang, 2010). Portfolio selection is very important for investors to get a good profit. Therefore, they want to choose the optimal portfolio that will provide them with minimum risk and maximum return. Portfolio analysis is about finding the most desired group of securities to obtain, given the characteristics of each security (Elton et al., 2009). Portfolios can be created in different numbers by choosing various stocks from securities. According to Markowitz, the process of choosing a portfolio is two stages: The first phase begins with observation and experience and ends with beliefs about the future performance of existing securities. The second stage starts

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with relevant beliefs about future performances and ends with portfolio selection (Markowitz, 1952).

The problem of optimizing a portfolio is one of the most studied and classical topics in computational finance. The modern portfolio theory first began with the calculation of how to achieve the risk-to-maximum rate of return in Harry Markowitz's study "Portfolio Selection" published in 1952. With the Markowitz average-variance model, the relationship of assets with each other within the framework of risk-return exchange has been revealed, so diversification and evaluation of the entire portfolio has been brought to the agenda. The Mean Variance Theory, widely known in asset management industry, focuses on a single-period (batch) portfolio selection to trade off a portfolio's expected return and risk, which typically determines the optimal portfolios subject to the investor's risk-return profile (Lie and Hoi, 2014). Investors prefer a portfolio of securities with high expected returns. According to Markowitz, in addition to the expected return, the correlation between the stocks that make up the portfolio is also important. Because the low correlation between them indicates that stocks will also have different returns. This reduces the risk.

Markowitz model constitutes a mathematically constrained optimization problem that the portfolio variance problem to be minimized. This optimization problem is a multi-objectives optimization problems with non-linear constrains and have many solution methods. However, the non-linear constrained portfolio optimization problem with multi-objective functions cannot be efficiently solved using traditionally approaches (Zhu et al, 2011).

Constrained optimization consists of optimization problems with constraints in the form of equality or inequality. In constrained optimization problems, solution methods have been developed according to whether constraints are equality or inequality. The analytical solution of constrained optimization problems can be investigated by using Lagrange multipliers in cases where constraints are in the form of equality, and Kuhn-Tucker conditions in case of constraints inequality. These methods are also useful for finding the solution of the Markowitz mean variance problem since it is adaptable to computer programming languages by making a suitable modification. For the classical Markowitz mean-variance problem, Pardalos et al. (1994) did some preliminary computational results using Kuhn Tucker approach for constrained optimization. They showed that the dual algorithm used this method performs efficiently on this special class of problems although it is a general purpose algorithm.

In this study, two optimal portfolios are obtained from the stocks traded in Turkish BIST 30 index by using the Markowitz's modern portfolio theory. As of February 15th, 2020, we perform calculations according to the stocks in Turkish BIST 30 index. We take the arranged closing prices of end of the months from January 31th, 2017 to January 31th, 2020, from the iş investment's official website (iş investment, 2020). We calculate the statistical formulations, such as expected return, risk and correlation, etc., by means of the Excell. Multi-variable equations are obtained for both portfolios and created constrains for optimization problem. Weights of stocks are calculated by Kuhn-Tucker method. Obtained equations with non-linear constraints are solved by using Matlab.

2. Preliminaries

In this section, basic concepts and notations that will be needed in the next sections are included.

2.1. Markowitz's portfolio selection method

In Markowitz mean–variance model, the security selection of risky portfolio construction is considered as one objective function and the mean return is defined as one of the constraints (Zhu et al., 2011). In this subsection, briefly we give some information about this model.

In order to use the Markowitz model in portfolio creation, we need to find the expected return, standard deviation, variance, coefficient of variation and correlation values. According to Markowitz, it is not possible for the individual or institution that invests their money in securities to know how much they will earn as a result of this investment. However, investors can reach some results by making use of historical data of stocks. At this stage, the expected return of the security needs to be calculated. Based on historical data, expected return calculation is calculated by taking the arithmetic average of returns.

Now we assume that there are n securities denoted by S_i (i=1,…, n), the return of the security Sj is denoted as R_i and the proportion of total investment funds devoted to this security is denoted as X_i. The expected return μ is then calculated as follows

$$\mu_i = \frac{1}{n}\sum_{i=1}^{n} R_i$$

(İnan et al., 2013; Tanaka et al., 2000).

The returns of the stocks vary randomly. In a period of time, the stock, where the best profits are obtained, can also cause great losses in the following period. For this, besides the expected return in an investment, it is necessary to look at how far the returns differ from the average. Thus, it is necessary to calculate the standard deviation σ and then the variance (risk) defined as the square of the standard deviation σ^2. The standard deviation for a sample is calculated as follows,

$$\sigma = \sqrt{\sum_{i=1}^{n} \frac{(R_i - \mu_i)^2}{n-1}}$$

(Barlow, 1993). By dividing the standard deviation of a stock by its expected return, the coefficient of variation of that stock is obtained.

Another criterion in calculating the securities that will form the portfolio is correlation. The correlation shows the degree of the relationship between the two securities. We know that a correlation coefficient has a maximum value of +1 and a minimum value of -1. A value of +1 means that two securities will always move in perfect unison, whereas a value of -1 means that their movements are exactly opposite to each other (Elton et al., 2009). If there is a negative correlation between the securities in the portfolio, the risk of the portfolio is reduced and even the non-systematic risk can be completely eliminated depending on the weight of these securities in the portfolio. However, the periods of increase or decrease in BIST shares are close to each other. In other words, it is not possible for the correlation between them to be -1. It is even hard to find a negative correlation. Suppose that P_{ik} stand for the correlation between securities S_i and S_k. The correlation coefficient between S_i and S_k is defined as follows

$$P_{ik} = \frac{\sigma_{ik}}{\sigma_i \sigma_k},$$

where σ_{ik} is the covariance between two stocks, σ_i and σ_k are standard deviation of the corresponding stock. It is possible to measure the individual risks of securities by variance. However, when there are two or more securities issues, the risk is expressed in covariance (Markowitz, 1952).

The covariance is a measure of how returns on assets move together (Elton et al., 2009). The covariance between S_i and S_k can be expressed as

$$\sigma_{ik} = \sum_{j=1}^{n} \frac{(R_{ij} - \mu_i)(R_{kj} - \mu_k)}{n-1}.$$

The Markowitz average-variance model aims to find a portfolio with a minimum variance (minimum risk) to meet the expected return level.

Markowitz mean–variance model is described in the following.

$$min \sum_{i=1}^{n} X_i \sum_{j=1}^{n} X_j \, \sigma_{ij}$$

with the linear constrained

$$\sum_{i=1}^{n} X_i \mu_i \geq R$$

and

$$\sum_{i=1}^{n} X_i = 1$$

where n is the number of different assets, σ_{ij} is the covariance between returns of assets S_i and S_j, X_i , $0 \leq X_i \leq 1, i = 1, \ldots, n,$ is the weight of each stock in the portfolio, $X_i \mu_i$ is the mean return of stock S_i and R is the desired mean return of the portfolio. (Markowitz, 1952; Zhu et al., 2011).

2.2. What is Turkish BIST 30 index?

The Istanbul stock exchange has gathered all the exchanges operating in the Turkish capital markets under one roof. Thus, Borsa İstanbul A. Ş., mostly known with its abbreviation of BIST was registered on April 3, 2013 as a securities exchange of Turkey. There are three main equity indexes in Turkish stock market; BIST 100 (XU100), BIST 50 (XU050) and BIST 30 (XU030). BIST 30 index consists of 30 stocks selected among the stocks of companies traded on BIST Stars which is the market of companies whose value of traded shares in BIST 100 index and market value is equal or above TRY 100,000,000 (Karakurt, 2018; KAP, 2020).

The main reason why we study BIST-30 index is that the BIST-30 index is formed from among the stocks traded in the stock exchange, among those with high market value and trading volume, taking into account their sectoral representation capabilities (Sevinç, 2014). BIST-30 consists of the securities given in Table 2.1., which are traded continuously within the scope of BIST in the period of January 2017 - January 2020 (KAP, 2020).

Table 2.1. Stocks traded within the scope of BIST-30 as of the end of the 2nd quarter of 2020

CODE	**COMPANY NAME**	**CODE**	**COMPANY NAME**
AKBNK	AKBANK T.A.Ş.	SODA	SODA SANAYİİ A.Ş.
ARCLK	ARÇELİK A.Ş.	TAVHL	TAV HAVALİMANLARI HOLDİNG A.Ş.
ASELS	ASELSAN ELEKTRONİK SANAYİ VE TİCARET A.Ş.	TKFEN	TEKFEN HOLDİNG A.Ş.
BIMAS	BİM BİRLEŞİK MAĞAZALAR A.Ş.	TOASO	TOFAŞ TÜRK OTOMOBİL FABRİKASI A.Ş.
DOHOL	DOĞAN ŞİRKETLER GRUBU HOLDİNG A.Ş.	TCELL	TURKCELL İLETİŞİM HİZMETLERİ A.Ş.
EKGYO	EMLAK KONUT GAYRİMENKUL YATIRIM ORTAKLIĞI A.Ş.	TUPRS	TÜPRAŞ-TÜRKİYE PETROL RAFİNERİLERİ A.Ş.
FROTO	FORD OTOMOTİV SANAYİ A.Ş.	THYAO	TÜRK HAVA YOLLARI A.O.
EREGL	EREĞLİ DEMİR VE ÇELİK FABRİKALARI T.A.Ş.	TTKOM	TÜRK TELEKOMÜNİKASYON A.Ş.
SAHOL	HACI ÖMER SABANCI HOLDİNG A.Ş.	GARAN	TÜRKİYE GARANTİ BANKASI A.Ş.
KRDMD	KARDEMİR KARABÜK DEMİR ÇELİK SANAYİ VE TİCARET A.Ş.	HALKB	TÜRKİYE HALK BANKASI A.Ş.
KCHOL	KOÇ HOLDİNG A.Ş.	ISCTR	TÜRKİYE İŞ BANKASI A.Ş.
KOZAL	KOZA ALTIN İŞLETMELERİ A.Ş.	TSKB	TÜRKİYE SINAİ KALKINMA BANKASI A.Ş.
KOZAA	KOZA ANADOLU METAL MADENCİLİK İŞLETMELERİ A.Ş.	SISE	TÜRKİYE ŞİŞE VE CAM FABRİKALARI A.Ş.
PGSUS	PEGASUS HAVA TAŞIMACILIĞI A.Ş.	VAKBN	TÜRKİYE VAKIFLAR BANKASI T.A.O.
PETKM	PETKİM PETROKİMYA HOLDİNG A.Ş.	YKBNK	YAPI VE KREDİ BANKASI A.Ş.

3. Solution of the Constrained Optimization Problem

Motivated by Markowitz model, we first create two optimum portfolios based on the correlation coefficient between stocks traded on BIST 30. Then we create the constrained optimization problem for these portfolios for calculating minimum risk and maximum return. Finally, we solve this problem using Matlab and calculate the weights of stocks in the portfolio for maximum return.

3.1. Creating the optimal portfolio

We firstly want to create an optimal portfolio among the stocks traded on BIST 30 through the Markowitz model. By using the Excell, we find the expected return, standard deviation, variance and coefficient of variation. Since we do not include the stocks with negative expected returns in the portfolio, we do not calculate the standard deviation of those stocks Results of stocks with positive expected returns are obtained as follows in Table 3.1.

Table 3.1. BIST 30 Expected return and risk table of stocks

	Expected Value	**Standard Deviation**	**Variance (Risk)**	**Coefficient of Variation**
AKBANK	0,0107	0,1018	0,0104	9,5327
ARCLK	0,0032	0,0896	0,0080	27,7728
ASELSAN	0,0203	0,1005	0,0101	4,9546
BIMAS	0,0207	0,0590	0,0035	2,8471
DOHOL	0,0365	0,1494	0,0223	4,0892
EREGL	0,0273	0,1050	0,0110	3,8508
FROTO	0,0299	0,0836	0,0070	2,7952
GARANT	0,0174	0,1061	0,0113	6,1157
ISCTR	0,0127	0,1011	0,0102	7,9761
KCHOL	0,0119	0,0788	0,0062	6,6441
KOZAA	0,0661	0,2161	0,0467	3,2687
KOZAL	0,0507	0,1375	0,0189	2,7131
KRDMD	0,0344	0,1340	0,0180	3,9012
PETKM	0,0176	0,1034	0,0107	5,8826
PGSUS	0,0554	0,1736	0,0302	3,1370
SAHOL	0,0055	0,0881	0,0078	16,0336
SISE	0,0190	0,1012	0,0102	5,3215
SODA	0,0188	0,0869	0,0075	4,6114
TAVHL	0,0256	0,1008	0,0102	3,9345
TCELL	0,0150	0,0770	0,0059	5,1181
THYAO	0,0337	0,1316	0,0173	3,9026
TKFEN	0,0355	0,1082	0,0117	3,0467
TOASO	0,0082	0,0808	0,0065	9,8934
TSKB	0,0129	0,0924	0,0085	7,1465
TTKOM	0,0147	0,1092	0,0119	7,4304
TUPRS	0,0208	0,0846	0,0072	4,0649
VAKBN	0,0157	0,1204	0,0145	7,6468
YKBANK	0,0104	0,1038	0,0108	10,0205

Now, it is tried to choose the stocks that have the least relationship between them. Among the stocks with the lowest expected return and the highest coefficient of variation, no covariance and correlation coefficient are calculated, that is, they are not included in the optimal portfolio. The covariance matrix for choosing optimal portfolio is obtained by using Excell and given by the following Table 3.2.

Table 3.2. The obtained covariance matrix for choosing optimal portfolio.

COV	KOZAA	PGSUS	KOZAL	DOHOL	TKFEN	KRDMD	THYAO	FROTO	EREGL	TAVHL	TUPRS	BIMAS	ASELSAN	SISE	SODA	PETKM	GARANT	VAKBN	TCELL	TSKB
KOZAA	0,0467																			
PGSUS	0,0166	0,0302																		
KOZAL	0,0203	0,0119	0,0189																	
DOHOL	0,0058	0,0037	0,0069	0,0223																
TKFEN	0,0010	0,0049	0,0019	-0,0010	0,0117															
KRDMD	0,0133	0,0086	0,0071	0,0056	0,0048	0,0180														
THYAO	0,0105	0,0101	0,0038	0,0003	0,0059	0,0080	0,0173													
FROTO	0,0025	0,0049	0,0019	0,0024	0,0035	0,0060	0,0032	0,0070												
EREGL	0,0039	0,0035	0,0007	0,0003	0,0052	0,0101	0,0061	0,0057	0,0110											
TAVHL	0,0042	0,0083	0,0036	0,0041	0,0053	0,0065	0,0079	0,0041	0,0053	0,0102										
TUPRS	0,0026	0,0009	0,0008	0,0027	0,0017	0,0030	0,0012	0,0016	0,0024	0,0010	0,0072									
BIMAS	0,0048	0,0035	0,0022	0,0008	0,0022	0,0019	0,0036	0,0013	0,0007	0,0015	0,0024	0,0035								
ASELSAN	0,0027	0,0056	0,0018	0,0025	0,0036	0,0033	0,0069	0,0030	0,0034	0,0033	0,0035	0,0030	0,0101							
SISE	0,0006	0,0038	0,0021	0,0055	0,0057	0,0053	0,0060	0,0030	0,0049	0,0072	0,0022	0,0021	0,0049	0,0102						
SODA	0,0004	-0,0033	-0,0008	0,0049	0,0021	0,0015	0,0008	0,0012	0,0030	0,0035	0,0013	0,0002	0,0018	0,0055	0,0075					
PETKM	-0,0004	0,0035	0,0027	0,0043	0,0045	0,0043	0,0044	0,0033	0,0028	0,0050	0,0026	0,0020	0,0045	0,0069	0,0027	0,0107				
GARANT	0,0075	0,0073	0,0041	0,0001	0,0020	0,0049	0,0060	0,0022	0,0029	0,0003	0,0028	0,0029	0,0051	0,0025	-0,0017	0,0032	0,0113			
VAKBN	0,0068	0,0065	0,0038	0,0004	0,0026	0,0071	0,0053	0,0021	0,0046	0,0005	0,0029	0,0033	0,0054	0,0040	-0,0010	0,0045	0,0108	0,0145		
TCELL	0,0013	0,0072	0,0032	0,0002	0,0040	0,0018	0,0045	0,0015	0,0011	0,0019	0,0008	0,0017	0,0043	0,0025	-0,0016	0,0027	0,0050	0,0052	0,0059	
TSKB	0,0051	0,0059	0,0031	0,0003	0,0009	0,0043	0,0057	0,0014	0,0029	0,0011	0,0009	0,0023	0,0040	0,0028	-0,0013	0,0029	0,0080	0,0093	0,0038	0,0085

By using the corresponding value of covariance, we obtained the correlation matrix between stocks as follows Table 3.3.

Table 3.3. The obtained correlation matrix for choosing optimal portfolio.

COR	KOZAA	PGSUS	KOZAL	DOHOL	TKFEN	KRDMD	THYAO	FROTO	EREGL	TAVHL	TUPRS	BIMAS	ASELSAN	SISE	SODA	PETKM	GARANT	VAKBN	TCELL	TSKB
KOZAA	1,0000																			
PGSUS	0,4433	1,0000																		
KOZAL	0,6833	0,4999	1,0000																	
DOHOL	0,1802	0,1410	0,3380	1,0000																
TKFEN	*0,0436*	0,2600	0,1290	-0,0589	1,0000															
KRDMD	0,4605	0,3680	0,3861	0,2790	0,3313	1,0000														
THYAO	0,3678	0,4427	0,2117	*0,0142*	0,4132	0,4543	1,0000													
FROTO	0,1368	0,3398	0,1679	0,1928	0,3860	0,5351	0,2900	1,0000												
EREGL	0,1722	0,1910	*0,0456*	*0,0196*	0,4582	0,7201	0,4445	0,6511	1,0000											
TAVHL	0,1933	0,4713	0,2603	0,2727	0,4817	0,4814	0,5988	0,4888	0,5025	1,0000										
TUPRS	0,1400	*0,0608*	*0,0713*	0,2144	0,1890	0,2611	0,1034	0,2326	0,2711	0,1155	1,0000									
BIMAS	0,3767	0,3446	0,2739	*0,0861*	0,3449	0,2411	0,4579	0,2536	0,1126	0,2557	0,4856	1,0000								
ASELSAN	0,1264	0,3182	0,1296	0,1692	0,3332	0,2484	0,5199	0,3518	0,3230	0,3216	0,4064	0,5105	1,0000							
SISE	*0,0289*	0,2140	0,1510	0,3660	0,5199	0,3875	0,4531	0,3587	0,4648	0,7083	0,2521	0,3471	0,4790	1,0000						
SODA	*0,0219*	-0,2161	-0,0666	0,3743	0,2199	0,1306	*0,0690*	0,1716	0,3299	0,4035	0,1802	*0,0484*	0,2046	0,6211	1,0000					
PETKM	-0,0176	0,1970	0,1878	0,2779	0,4041	0,3094	0,3203	0,3820	0,2534	0,4789	0,2988	0,3321	0,4281	0,6584	0,3050	1,0000				
GARANT	0,3251	0,3955	0,2795	*0,0061*	0,1723	0,3471	0,4275	0,2440	0,2571	*0,0255*	0,3106	0,4628	0,4805	0,2330	-0,1836	0,2880	1,0000			
VAKBN	0,2615	0,3110	0,2277	*0,0230*	0,2005	0,4381	0,3339	0,2075	0,3667	*0,0437*	0,2884	0,4620	0,4494	0,3262	-0,0960	0,3647	0,8486	1,0000		
TCELL	*0,0751*	0,5368	0,2978	*0,0159*	0,4839	0,1765	0,4436	0,2301	0,1406	0,2492	0,1239	0,3717	0,5617	0,3269	-0,2371	0,3429	0,6133	0,5631	1,0000	
TSKB	0,2546	0,3692	0,2409	*0,0184*	*0,0938*	0,3449	0,4694	0,1857	0,2941	0,1230	0,1113	0,4302	0,4337	0,3023	-0,1568	0,3021	0,8148	0,8357	0,5319	1,0000

If the correlation between the two stocks is 1, this means that the two companies earn or lose at approximately the same rate. In other words, it can be interpreted that it is not an advantage to include the two in the same portfolio. Therefore, in the next step, we will include stocks with minimum correlation between stocks in the portfolio to minimize risk. In general, a portfolio of BIST 30 stocks with high expected return, low coefficient of variation and less than 41 percent correlation between stocks is created as follows in Table 3.4. The stocks included in the portfolio are KOZAL, DOHOL, TKFEN, FROTO, TUPRS. SODA, PETKM and TSKB.

Table 3.4. Correlation matrix of the created optimal portfolio with eight stocks

COR	KOZAL	DOHOL	TKFEN	FROTO	TUPRS	SODA	PETKM	TSKB
KOZAL	1,0000							
DOHOL	0,3380	1,0000						
TKFEN	0,1290	-0,0589	1,0000					
FROTO	0,1679	0,1928	0,3860	1,0000				
TUPRS	0,0713	0,2144	0,1890	0,2326	1,0000			
SODA	-0,0666	0,3743	0,2199	0,1716	0,1802	1,0000		
PETKM	0,1878	0,2779	0,4041	0,3820	0,2988	0,3050	1,0000	
TSKB	0,2409	0,0184	0,0938	0,1857	0,1113	-0,1568	0,3021	1,0000

Table 3.5 shows the correlation matrix created among portfolio of BIST 30 stocks with high expected return, low coefficient of variation and less than 25 percent correlation between stocks. The stocks included in the portfolio are KOZAL, FROTO, TUPRS, SODA and TSKB.

Table 3.5. Correlation matrix of the created optimal portfolio with five stocks

COR	KOZAL	FROTO	TUPRS	SODA	TSKB
KOZAL	1,0000				
FROTO	0,1679	1,0000			
TUPRS	0,0713	0,2326	1,0000		
SODA	-0,0666	0,1716	0,1802	1,0000	
TSKB	0,2409	0,1857	0,1113	-0,1568	1,0000

3.2. Solution of the optimization problem

In this subsection we will find the portfolio risk according to the Markowitz average variance model for the two portfolios created in the previous subsection. For convenience, we will show the weights of stocks with x_i, $0<i<9$, and y_j, $0<j<6$, values.

Table 3.6. The created optimal portfolios

OP-1:The created optimal portfolio with eight stocks ($P_{ik} < 0{,}41$)			OP-2:The created optimal portfolio with five stocks ($P_{ik} < 0{,}25$)		
Weights	Stocks	Expected return	Weights	Stocks	Expected return
x_1	KOZOL	0,0507	y_1	KOZOL	0,0507
x_2	DOHOL	0,0365	y_2	FROTO	0,0299
x_3	TKFEN	0,0355	y_3	TUPRS	0,0208
x_4	FROTO	0,0299	y_4	SODA	0,0188
x_5	TUPRS	0,0208	y_5	TSKB	0,0129
x_6	SODA	0,0188			
x_7	PETKM	0,0176			
x_8	TSKB	0,0129			

The covariance matrixes among the stocks that make up the portfolios is given below.

Table 3.7. The covariance matrixes of the created optimal portfolio with eight stocks

COV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_1	0,0189							
x_2	0,0069	0,0223						
x_3	0,0019	-0,0010	0,0117					
x_4	0,0019	0,0024	0,0035	0,0070				
x_5	0,0008	0,0027	0,0017	0,0016	0,0072			
x_6	-0,0008	0,0049	0,0021	0,0012	0,0013	0,0075		
x_7	0,0027	0,0043	0,0045	0,0033	0,0026	0,0027	0,0107	
x_8	0,0031	0,0003	0,0009	0,0014	0,0009	-0,0013	0,0029	0,0085

Table 3.8. The covariance matrixes of the created optimal portfolio with five stocks

COV	y_1	y_2	y_3	y_4	y_5
y_1	0,0189				
y_2	0,0019	0,0070			
y_3	0,0008	0,0016	0,0072		
y_4	-0,0008	0,0012	0,0013	0,0075	
y_5	0,0031	0,0014	0,0009	-0,0013	0,0085

By using the expected values of stocks in Table 3.6. and the covariances between stocks in Table 3.7 for OP-1 and Table 3.8 for OP-2, we obtain the portfolio variance (risk) according to Markowitz mean–variance model. Then we solve these optimization problem by Kuhn-Tucker method. A solution of a minimization problem which has the following model

$$\text{Objective function; } \min f(x_1, x_2, \ldots, x_n)$$

with constrains

$$g_1(x_1, x_2, \ldots x_n) \le b_1,\ g_2(x_1, x_2, \ldots x_n) \le b_2, \ldots, g_m(x_1, x_2, \ldots x_n) \le b_m$$

is found as follows:

Let $\dot{x} = (\dot{x}_1, \dot{x}_1, \ldots, \dot{x}_n)$ be a solution of the problem. $\dot{x}$ satisfies the objective function with all constrained and has the multipliers $\lambda_1, \lambda_2, \ldots \lambda_m$ hold by the following conditions

$$L(x_i, \lambda_i) = f(x_j) + \sum_{i=1}^{m} \lambda_i g_i(x_j),\quad j = 1,2, \ldots, n$$

$$\frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(x_j)}{\partial x_j} = 0,\ b_i - g_i \le 0,\ \ \lambda_i(b_i - g_i) = 0,\ \ \lambda_i \ge 0, = 1,2, \ldots, m$$

(Wallace, 2004). Then we respectively solve the problems OP-1 and OP-2 with the Kuhn-Tucker method. For OP-1, Constrained are found as

i. The model for OP-1

According to OP-1, the objective function obtain as follows

$minf\ (x_1, x_2, \ldots, x_n) = 0{,}0189x_1^2 + 0{,}0223x_2^2 + 0{,}0117\, x_3^2 + 0{,}007x_4^2 + 0{,}0072\, x_5^2 + 0{,}0075\, x_6^2 + 0{,}0107x_7^2 + 0{,}0085x_8^2 + 0{,}0138x_1x_2 + 0{,}0038x_1x_3 + 0{,}0038x_1x_4 + 0{,}0016x_1x_5 - 0{,}0016x_1x_6 + 0{,}0054x_1x_7 + 0{,}0062x_1x_8 - 0{,}002x_2x_3 + 0{,}048x_2x_4 + 0{,}0054x_2x_5 + 0{,}0098x_2x_6 + 0{,}0086x_2x_7 + 0{,}0006x_2x_8 + 0{,}007x_3x_4 + 0{,}0034x_3x_5 + 0{,}0042x_3x_6 + 0{,}009x_3x_7 + 0{,}0018x_3x_8 + 0{,}0032x_4x_5 + 0{,}0024x_4x_6 + 0{,}0066x_4x_7 + 0{,}0028x_4x_8 + 0{,}0026x_5x_6 + 0{,}0052x_5x_7 + 0{,}0018x_5x_8 + 0{,}0054x_6x_7 - 0{,}0026x_6x_8 + 0{,}0058x_7x_8$

with the linear constrains

$$-0{,}0507x_1 - 0{,}0365x_2 - 0{,}0355x_3 - 0{,}0299x_4 - 0{,}0208x_5 + 0{,}0188x_6 + 0{,}0176x_7 + 0{,}0129x_8 \le -0.0278,$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \le 1$$

and

$$-x_i \le 0, \qquad 1 \le x_i \le 8.$$

So, we have

$L(x_i, \lambda_i) = 0{,}0189x_1^2 + 0{,}0223x_2^2 + 0{,}0117\, x_3^2 + 0{,}007x_4^2 + 0{,}0072\, x_5^2 + 0{,}0075\, x_6^2 + 0{,}0107x_7^2 + 0{,}0085x_8^2 + 0{,}0138x_1x_2 + 0{,}0038x_1x_3 + 0{,}0038x_1x_4 + 0{,}0016x_1x_5 - 0{,}0016x_1x_6 + 0{,}0054x_1x_7 + 0{,}0062x_1x_8 - 0{,}002x_2x_3 + 0{,}048x_2x_4 + 0{,}0054x_2x_5 + 0{,}0098x_2x_6 + 0{,}0086x_2x_7 + 0{,}0006x_2x_8 + 0{,}007x_3x_4 + 0{,}0034x_3x_5 + 0{,}0042x_3x_6 + 0{,}009x_3x_7 + 0{,}0018x_3x_8 + 0{,}0032x_4x_5 + 0{,}0024x_4x_6 + 0{,}0066x_4x_7 + 0{,}0028x_4x_8 + 0{,}0026x_5x_6 + 0{,}0052x_5x_7 + 0{,}0018x_5x_8 + 0{,}0054x_6x_7 - 0{,}0026x_6x_8 + 0{,}0058x_7x_8 + \lambda_1(-0{,}0507x_1 - 0{,}0365x_2 -$

0,0355x_3 - 0,0299x_4 -0,0208x_5 $-0.0188x_6$ - 0,0176x_7 -0,0129x_8) + λ_2 (x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8)- $\lambda_3 x_1$-$\lambda_4 x_2$ - $\lambda_5 x_3$ - $\lambda_6 x_4$ - $\lambda_7 x_5$ - $\lambda_8 x_6$ - $\lambda_9 x_7$ - $\lambda_{10} x_8$.

For solving this problem, we obtain first derivatives

$\frac{\partial L}{\partial x_1}$=0,0378$x_1$+ 0,0138$x_2$ + 0,0038x_3 + 0,0038x_4 +0,0016 x_5 - 0,0016x_6 + 0,0054x_7 +0,0062 x_8

-0,0507 λ_1 + λ_2 -λ_3 =0

$\frac{\partial L}{\partial x_2}$=0,0446$x_2$+ 0,0138$x_1$ - 0,002x_3 + 0,048x_4 +0,0054 x_5 +0,0098x_6 + 0,0086x_7 +0,0006 x_8

-0,0365 λ_1 + λ_2 -λ_4 =0

$\frac{\partial L}{\partial x_3}$=0,0234$x_3$+ 0,0038$x_1$ - 0,002x_2 + 0,007x_4 +0,0034 x_5 +0,0042x_6 + 0,009x_7 +0,0018 x_8

-0,0355 λ_1 + λ_2 -λ_5 =0

$\frac{\partial L}{\partial x_4}$=0,014$x_4$+ 0,0038$x_1$ + 0,0482x_2 + 0,007x_3 +0,0032 x_5 +0,0024x_6 + 0,0066x_7 +0,0028 x_8

-0,0299 λ_1 + λ_2 -λ_6 =0

$\frac{\partial L}{\partial x_5}$=0,0144$x_5$+ 0,0016$x_1$ + 0,0054x_2 + 0,0034x_3 +0,0032 x_4 +0,0026x_6 + 0,0052x_7 +0,0018 x_8

-0,0208 λ_1 + λ_2 -λ_7 =0

$\frac{\partial L}{\partial x_6}$=0,015$x_6$- 0,0016$x_1$ + 0,0098x_2 + 0,0042x_3 +0,0024 x_4 +0,0026x_5 + 0,0054x_7 - 0,0026 x_8

- 0,0188 λ_1 + λ_2 -λ_8 =0

$\frac{\partial L}{\partial x_7}$=0,00214$x_7$+ 0,0054$x_1$ + 0,0086x_2 + 0,009x_3 +0,0066 x_4 +0,0052x_5 + 0,0054x_6 +0,0058 x_8

- 0,0176 λ_1 + λ_2 -λ_9 =0

$\frac{\partial L}{\partial x_8}$=0,017$x_8$+ 0,0062$x_1$ + 0,0006x_2 + 0,0018x_3 +0,0028 x_4 +0,0018x_5 - 0,0026x_6 +0,0058 x_7

- 0,0129 λ_1 + λ_2 -λ_{10} =0

and constraints

λ_1(-0,0278 + 0,0507x_1 + 0,0365x_2 + 0,0355x_3 +0,0299x_4 +0,0208x_5 + 0,0188x_6 + 0,0176x_7 +0,0129x_8)=0

λ_2 (1- x_1 - x_2 - x_3 - x_4 -x_5 - x_6 - x_7 -x_8)=0

$\lambda_3(x_1)$ =0 $\lambda_4(x_2)$ =0 $\lambda_5(x_3)$ =0 $\lambda_6(x_4)$ =0 $\lambda_7(x_5)$ =0 $\lambda_8(x_6)$ =0 $\lambda_9(x_7)$ =0 $\lambda_{10}(x_8)$ =0.

Choosing $\lambda_3 = \lambda_5 = \lambda_6 = \lambda_9 = \lambda_{10} = 0$, we solve this equation by means of the following Matlab codes

Q=[0.0378 0.0138 0.0038 0.0038 0.0016 -0.0016 0.0054 0.0062 -0.0507 1 -1 0 0 0 0 0 0 0;0.0446 0.0138 -0.002 0.048 0.0054 0.0098 0.0086 0.0006 -0.0365 1 0 -1 0 0 0 0 0 0; 0.0234 0.0038 -0.002 0.007 0.0034 0.0042 0.009 0.0018 -0.0355 1 0 0 -1 0 0 0 0 0; 0.014 0.0038 0.0482 0.007 0.0032 0.0024 0.0066 0.0028 -0.0299 1 0 0 0 -1 0 0 0 0;0.0144 0.0016 0.0054 0.0034 0.0032 0.0026 0.0052 0.0018 -0.0208 1 0 0 0 0 -1 0 0 0; 0.015 -0.0016 0.0098 0.0042 0.0024 0.0026 0.0054 -0.0026 -0.0188 1 0 0 0 0 0 -1 0 0;0.00214 0.0054 0.0086 0.009 0.0066 0.0052 0.0054 0.0058 -0.0176 1 0 0 0 0 0 0 -1 0; 0.017 0.0062 0.0006 0.0018 0.0028 0.0018 -0.0026 0.0058 -0.0129 1 0 0 0 0 0 0 0 -1; 0.0507 0.0365 0.0355 0.0299 0.0208 0.0188 0.0176 0.0129 0 0 0 0 0 0 0 0 0 0; -1 -1 -1 -1 -1 -1 -1 -1 0 0 0 0 0 0 0 0 0 0;0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0;0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0;0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0;0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0;0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0;0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0;0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0;0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1];

u=[0;0;0;0;0;0;0;0;0.0279;-1;0;0;0;0;0;0;0;0];

v=Q\u.

The solutions are found as

$x_{1=}$0,2106, $x_3 = 0{,}0228$ $x_4 = 0.3379$ $x_7 = 0.1663$ $x_8 = 0{,}2625, x_2 = x_5 = x_6 = 0,$

$\lambda_1 = 0{,}1736, \lambda_2 = -0{,}0031, \lambda_4 0{,}0178, \lambda_7 = -0{,}0010, \lambda_8 = -0{,}0013.$

Different results can be produced with the codes given for different selections of the λ_i.

ii. The model for OP-2

According to OP-2, the objective function obtain as follows

minh($y_1, y_2, \dots y_5$)=0,0189y_1^2+ 0,007y_2^2 + 0,0072 y_3^2 + 0,0075y_4^2 + 0,0085 y_5^2 + 0,0038 $y_1 y_2$ +0,0016$y_1 y_3$ - 0,0016$y_1 y_4$ + 0,0062$y_1 y_5$ + 0,0032$y_2 y_3$ + 0,0024$y_2 y_4$+ 0,0028$y_2 y_5$ + 0,0026$y_3 y_4$ +0,0018$y_3 y_5$ - 0,0026$y_4 y_5$

with the linear constrains

-0,0507y_1- 0,0299y_2 - 0,0208y_3 - 0,0188y_4 -0,0129y_5 $\leq$ 0.0266,

$y_1 + y_2 + y_3 + y_4 + y_5 \leq 1$

and

-$y_i \leq 0$, $1 \leq y_i \leq 5$.

So, we have

L(y_j, λ_i) = 0,0189y_1^2+ 0,007y_2^2 + 0,0072 y_3^2 + 0,0075y_4^2 + 0,0085 y_5^2 + 0,0038 $y_1 y_2$ +0,0016$y_1 y_3$ - 0,0016$y_1 y_4$ + 0,0062$y_1 y_5$ + 0,0032$y_2 y_3$ + 0,0024$y_2 y_4$+ 0,0028$y_2 y_5$ + 0,0026$y_3 y_4$ +0,0018$y_3 y_5$ - 0,0026$y_4 y_5$ +λ_1(- 0,0507y_1- 0,0299y_2 - 0,0208y_3 - 0,0188y_4 - 0,0129y_5) + λ_2 (y_1+y_2+y_3+y_4+y_5)- $\lambda_3 y_1$-$\lambda_4 y_2$ - $\lambda_5 y_3$ - $\lambda_6 y_4$ - $\lambda_7 y_5$.

For solving this problem, we obtain first derivatives

$\frac{\partial L}{\partial y_1}$ =0,0378y_1+ 0,0038y_2 + 0,0016y_3 - 0,0016y_4 +0,0062 y_5 - 0,0507 λ_1 + λ_2 -λ_3 =0

$\frac{\partial L}{\partial y_2}$ =0,014y_2+ 0,0038y_1 + 0,0032y_3 + 0,0024y_4 +0,0028 y_5 - 0,0299 λ_1 + λ_2 -λ_4 =0

$\frac{\partial L}{\partial y_3}$ =0,0144y_3+ 0,0016y_1 + 0,0032y_2 +0,0026y_4 +0,0018 y_5 - 0,0208 λ_1 + λ_2 -λ_5 =0

$\frac{\partial L}{\partial y_4}$ =0,015y_4- 0,0016y_1 + 0,0024y_2 +0,0026y_3 -0,0026 y_5 - 0,0188 λ_1 + λ_2 -λ_6 =0

$\frac{\partial L}{\partial y_5}$ =0,017y_5+ 0,0062y_1 + 0,0028y_2 +0,0018y_3-0,0026 y_4 - 0,0129 λ_1 + λ_2 -λ_7 =0

λ_1(-0,0266 + 0,0507y_1 + 0,0299y_2 + 0,0208y_3 +0,0188y_4 +0,0129y_5) =0

and constraints

λ_2 (1- y_1 - y_2 - y_3 - y_4 -y_5)=0

$\lambda_3(y_1)$ =0 $\lambda_4(y_2)$ =0 $\lambda_5(y_3)$ =0 $\lambda_6(y_4)$ =0 $\lambda_7(y_5)$ =0.

Choosing $\lambda_3 = \lambda_5 = \lambda_6 = 0$ and by using Matlab with same codes above, we find $y_{1=}$0,2190, $y_3 = 0{,}4069, y_4 = 0.3741,\ y_2 = y_5 = 0\ \lambda_1 = 0{,}0971, \lambda_2 = -0{,}0034, \lambda_4 = -0{,}0034, \lambda_7 = 0{,}0009$.

Conclusions

We firstly aim to create optimal portfolios according to the Markowitz (modern) Portfolio Method using the historical data of the companies with Turkish BIST 30 index between January 31th, 2017 to January 31th, 2020. Taking into consideration the arithmetic average of the expected returns of the companies and correlations, we create two portfolio variance (portfolio risk) functions obtained according to the Markowitz Average Variance Model. Generated non-linear functions are constrained optimization problems with linear constraints. Then, these problems are solved by the Kuhn-Tucker method. In the following Table 3.9, we give the solutions

OP-1:The created optimal portfolio with eight stocks ($P_{ik} < 0{,}41$)			OP-2:The created optimal portfolio with five stocks ($P_{ik} < 0{,}25$)		
Weights	Stocks	Investment shares	Weights	Stocks	Investment shares
x_1	KOZOL	%21	y_1	KOZOL	%22
x_2	DOHOL	0	y_2	FROTO	0
x_3	TKFEN	%2	y_3	TUPRS	%41
x_4	FROTO	%34	y_4	SODA	%37
x_5	TUPRS	0	y_5	TSKB	0
x_6	SODA	0			
x_7	PETKM	%17			
x_8	TSKB	%26			

If these values are written into the relevant objective functions, the return variance i.e. risks of OP-1 and OP-2 are found as %0.43 and %0.35, respectively.

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